Discrete Consumption-Saving Choice

Problem:

$$\underset{c_t}{Max}\ E\sum_{t=0}^{\infty}\beta^t U(c_t)$$

subject to:

$$c_t + a_{t+1} = (1+r)a_t + ws_t$$

Random variable, s_t:

$$s_t = 0 : \text{ unemployed } (u)$$

$$s_t = 1 : \text{ unemployed } (e)$$

State Transition Matrix:

$$p = \begin{bmatrix} pr(s_{t+1} = 0|s_t = 0) & pr(s_{t+1} = 1|s_t = 0) \\ pr(s_{t+1} = 0|s_t = 1) & pr(s_{t+1} = 1|s_t = 1) \end{bmatrix}$$

Discrete Control Space $A : \{a_1, a_2, \cdots, a_n\}$

Bellman Equation:

$$v(a_t, s_t) = \underset{a_{t+1}}{Max}\ \{U((1+r)a_t + ws_t - a_{t+1}) + \beta E_t v(a_{t+1}, s_{t+1})\}$$

There are two employment states: u and e.

There are n asset states: $\{a_1, a_2, \cdots, a_n\}$.

There are $2n$ total state combinations.

Write the Bellman equation for all possible combinations.

Convention:

Rows correspond to initial states.

Columns correspond to choice possibilities.

Example: 3 asset states, a_1, a_2, a_3.

Define the quantity:

$$R = R(a_t, a_{t+1}, s_t) \equiv U((1+r)a_t + ws_t - a_{t+1})$$

Compose the matrix, $R1$:

$$R1 = \begin{bmatrix} R(a_1,a_1,u) & R(a_1,a_2,u) & R(a_1,a_3,u) \\ R(a_2,a_1,u) & R(a_2,a_2,u) & R(a_2,a_3,u) \\ R(a_3,a_1,u) & R(a_3,a_2,u) & R(a_3,a_3,u) \end{bmatrix}$$

Compose the matrix, $R2$:

$$R2 = \begin{bmatrix} R(a_1,a_1,e) & R(a_1,a_2,e) & R(a_1,a_3,e) \\ R(a_2,a_1,e) & R(a_2,a_2,e) & R(a_2,a_3,e) \\ R(a_3,a_1,e) & R(a_3,a_2,e) & R(a_3,a_3,e) \end{bmatrix}$$

Finally, define the matrix, R :

$$R = \begin{bmatrix} R1 \\ R2 \end{bmatrix} = R(a_t, a_{t+1})$$

Bellman equation:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \equiv \begin{bmatrix} v_1(a_1) \\ v_1(a_2) \\ v_1(a_3) \\ v_2(a_1) \\ v_2(a_2) \\ v_2(a_3) \end{bmatrix} \equiv \begin{bmatrix} v(a_1,u) \\ v(a_2,u) \\ v(a_3,u) \\ v(a_1,e) \\ v(a_2,e) \\ v(a_3,e) \end{bmatrix}$$

$$= \max_{a_{t+1}} \{R + \beta E v(a_{t+1}, s_{t+1} | s_t)\},$$

where:

$$\beta E v(a_{t+1}, s_{t+1} | s_t)$$

$$= \beta \begin{bmatrix} p_{11}v(a_1,u)+p_{12}v(a_1,e) & p_{11}v(a_2,u)+p_{12}v(a_2,e) & p_{11}v(a_3,u)+p_{12}v(a_3,e) \\ p_{11}v(a_1,u)+p_{12}v(a_1,e) & p_{11}v(a_2,u)+p_{12}v(a_2,e) & p_{11}v(a_3,u)+p_{12}v(a_3,e) \\ p_{11}v(a_1,u)+p_{12}v(a_1,e) & p_{11}v(a_2,u)+p_{12}v(a_2,e) & p_{11}v(a_3,u)+p_{12}v(a_3,e) \\ p_{21}v(a_1,u)+p_{22}v(a_1,e) & p_{21}v(a_2,u)+p_{22}v(a_2,e) & p_{12}v(a_3,u)+p_{22}v(a_3,e) \\ p_{21}v(a_1,u)+p_{22}v(a_1,e) & p_{21}v(a_2,u)+p_{22}v(a_2,e) & p_{12}v(a_3,u)+p_{22}v(a_3,e) \\ p_{21}v(a_1,u)+p_{22}v(a_1,e) & p_{21}v(a_2,u)+p_{22}v(a_2,e) & p_{12}v(a_3,u)+p_{22}v(a_3,e) \end{bmatrix}$$

Bellman equation may be written as:

$$\begin{bmatrix} v(a_1,u) \\ v(a_2,u) \\ v(a_3,u) \\ v(a_1,e) \\ v(a_2,e) \\ v(a_3,e) \end{bmatrix} = \max_{a_{t+1}} \left\{ R(a_t, a_{t+1}) + \beta \begin{bmatrix} p_{11} & p_{12} \\ p_{11} & p_{12} \\ p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{21} & p_{22} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} v_1(a_1) & v_1(a_2) & v_1(a_3) \\ v_2(a_1) & v_2(a_2) & v_2(a_3) \end{bmatrix} \right\}$$

The matrix of repeated p's may be coded as:

kron(p,ones(3,1))

The coding for the RHS of the Bellman equation is:

A=R+βkron(p,ones(3,1))*[v1';v2']

Value Function Iteration

We start with a guess of the value function, possibly:

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v'_{01} = [\ 0 \quad 0 \quad 0\ ] \\ v'_{02} = [\ 0 \quad 0 \quad 0\ ] \end{matrix}$$

In the above expression, A:

Rows correspond to states.

Columns correspond to choices.

To use the max command, we therefore need A'.

The command,

[v,k]=max(A')

produces:

v: the transpose of the maximized v, given the original v.

h:a row vector of the index numbers of the best choices given the original v.

We can iterate to get close to the correct value function.

Howard's Policy Improvement

We start with a policy guess.

Denote a policy, h, as:

$$h = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \\ h_{31} & h_{32} \end{bmatrix}$$

h_{ij} denotes policy for $a_t = a_i$ and $s_t = s_j$

We may also denote h as:

$$h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} h_{11} \\ h_{21} \\ h_{31} \\ h_{12} \\ h_{22} \\ h_{32} \end{bmatrix}$$

For given h, we must compute v.
Let R^h denote:

$$R^h = \begin{bmatrix} R(a_1, h(a_1, u), u) \\ R(a_2, h(a_2, u), u) \\ R(a_3, h(a_3, u), u) \\ R(a_1, h(a_1, e), e) \\ R(a_2, h(a_2, e), e) \\ R(a_3, h(a_3, e), e) \end{bmatrix}$$

We may now compute the value function $v(h)$ from the relationship:

$$\begin{bmatrix} v(a_1, u) \\ v(a_2, u) \\ v(a_3, u) \\ v(a_1, e) \\ v(a_2, e) \\ v(a_3, e) \end{bmatrix} = R^h + \beta E \begin{bmatrix} v(h(a_1, u), u) \\ v(h(a_2, u), u) \\ v(h(a_3, u), u) \\ v(h(a_1, e), e) \\ v(h(a_2, e), e) \\ v(h(a_3, e), e) \end{bmatrix}$$

To facilitate the computation, define the matrices, $J1(h)$ and $J2(h)$.
$J1$ corresponds to $s_t = 0$ (unemployed).

$J1$ elements are 1 in each row in the column to be chosen by the relevant h.

$$J1(i,j) = 1 \text{ if } h(a_i) = a_j$$

$$J1(i,j) = 0 \text{ if } h(a_i) \neq a_j$$

Example:

$$\text{For } h(s_t = 0) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$J1(h) \equiv \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$J2$ corresponds to $s_t = 1$ (employed).
$J2$ elements are 1 in each row in the column to be chosen by the relevant h.

$$J2(i,j) = 1 \text{ if } h(a_i) = a_j$$

$$J2(i,j) = 0 \text{ if } h(a_i) \neq a_j$$

The equation for the v's may now be written as:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v(a_1,u) \\ v(a_2,u) \\ v(a_3,u) \\ v(a_1,e) \\ v(a_2,e) \\ v(a_3,e) \end{bmatrix} = R^h + \beta \begin{bmatrix} p_{11}J1 & p_{12}J2 \\ p_{21}J2 & p_{22}J2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Solving, we get:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left(I - \beta \begin{bmatrix} p_{11}J1 & p_{12}J2 \\ p_{21}J2 & p_{22}J2 \end{bmatrix} \right)^{-1} \begin{bmatrix} R_1^h \\ R_2^h \end{bmatrix}$$

We next compute the optimal policy for the resulting values in v.
Same procedure as in the value function iteration.