Alternate Derivation of Q, R, W.

Consider a one-period reward function of the form

$$r = -\tfrac{1}{2}\left(\text{linear combination of } x\text{'s and } u\text{'s}\right)^2$$

Define $z = \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ u_1 \\ \vdots \\ u_k \end{bmatrix}$

"Linear combination" will be of the form

$$D \cdot z$$

where D is a row vector of the coefficients of the x and u terms

We are interested in

$$r = -\tfrac{1}{2}\left(D \cdot z\right)^2$$

But Dz is a scalar so $Dz = z'D'$

$$r = -\tfrac{1}{2}\left(z'D'Dz\right)$$

Calculate $D'D$ as needed from the problem.

Then we will want

$$r() = x_t' Q x_t + u_t' R u + 2x_t' W u_t$$

Therefore

$$\begin{bmatrix} Q & W \\ W' & R \end{bmatrix} = -\frac{1}{2} D'D$$

In the class example

$$r = -\frac{1}{2}\left(A_t - \frac{u_t}{R} - h_t - b\right)^2$$

and $x_t = \begin{bmatrix} 1 \\ h_t \\ A_t \end{bmatrix}$ $\quad u_t = u_t$

Therefore $z_t = \begin{bmatrix} 1 \\ h_t \\ A_t \\ u_t \end{bmatrix}$

$$Dz = \begin{bmatrix} -b & -1 & 1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} 1 \\ h_t \\ A_t \\ u_t \end{bmatrix}$$

$$D'D = \begin{bmatrix} -b \\ -1 \\ 1 \\ -\frac{1}{R} \end{bmatrix} \begin{bmatrix} -b & -1 & 1 & -\frac{1}{R} \end{bmatrix}$$

$$D'D = \begin{bmatrix} b^2 & b & -b & \frac{b}{R} \\ b & 1 & -1 & \frac{1}{R} \\ -b & -1 & 1 & -\frac{1}{R} \\ \frac{b}{R} & \frac{1}{R} & -\frac{1}{R} & \frac{1}{R^2} \end{bmatrix}$$

$$\begin{bmatrix} Q & \omega \\ \omega' & R \end{bmatrix} = \begin{bmatrix} -\frac{b^2}{2} & -\frac{b}{2} & \frac{b}{2} & -\frac{b}{2R} \\ -\frac{b}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2R} \\ \frac{b}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2R} \\ -\frac{b}{2R} & -\frac{1}{2R} & \frac{1}{2R} & -\frac{1}{2R^2} \end{bmatrix}$$

$$Q = \begin{bmatrix} -\frac{b^2}{2} & -\frac{b}{2} & \frac{b}{2} \\ -\frac{b}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{b}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad \omega = \begin{bmatrix} -\frac{b}{2R} \\ -\frac{1}{2R} \\ \frac{1}{2R} \end{bmatrix}$$

$$R = -\frac{1}{2R^2}$$