Total, Average, and Marginal Physical Products

Hold all but one of the inputs fixed (say, fix $K = \bar{K}$). Perhaps we are in a short run situation, or perhaps we are just focusing on the effect of changing $L$.

The total product of labor is given by the function, $x = f(L; \bar{K})$. We can graph this as a cross-section of the production function.

Cobb-Douglas Production Function
The average product of labor is defined as

$$AP_L = \frac{f(L; K)}{L}.$$ 

The marginal product of labor is defined as

$$MP_L = \frac{\partial f(L; K)}{\partial L}.$$ 

Cobb-Douglas example: \( x = K^\alpha L^\beta \)

\[
\begin{align*}
AP_L &= \frac{K^\alpha L^\beta}{L} = K^\alpha L^{\beta-1} \\
MP_L &= \frac{\partial K^\alpha L^\beta}{\partial L} = \beta K^\alpha L^{\beta-1} \\
AP_K &= \frac{K^\alpha L^\beta}{K} = K^{\alpha-1} L^\beta \\
MP_K &= \frac{\partial K^\alpha L^\beta}{\partial K} = \alpha K^{\alpha-1} L^\beta
\end{align*}
\]
Connection between MRTS and Marginal Products

We can get an expression for the MRTS by treating output as fixed and totally differentiating the equation

\[ f(K, L) = \bar{x}. \]

By holding \( x \) fixed, we are remaining on the same isoquant as we vary \( K \) and \( L \). This yields

\[ \frac{\partial f}{\partial L} dL + \frac{\partial f}{\partial K} dK = 0. \]

Rearranging, we have

\[ \frac{dK}{dL} \bigg|_{f(K,L) = \bar{x}} = \frac{\frac{\partial f}{\partial L}}{\frac{\partial f}{\partial K}}. \tag{1} \]

The left side of (1) is the MRTS, and the right side is the ratio of marginal products.

\[ MRTS = \frac{MP_L}{MP_K}. \tag{2} \]
Diminishing Marginal Returns

Diminishing marginal returns (to labor) occur when the marginal product (of labor) eventually falls as L increases.

\[
\frac{\partial MP_L}{\partial L} < 0
\]

That is, labor is less and less productive at the margin, as L increases. It can be shown that with CRS or DRS, we must have diminishing marginal returns to each input.

Cobb-Douglas example: \( x = K^\alpha L^\beta \)

\[
MP_L = \frac{\partial K^\alpha L^\beta}{\partial L} = \beta K^\alpha L^{\beta-1}
\]

\[
\frac{\partial MP_L}{\partial L} = (\beta - 1)\beta K^\alpha L^{\beta-2}
\]

Thus, we have diminishing marginal returns to labor when \( \beta < 1 \). Constant returns to scale and diminishing marginal returns can easily coexist.