Bayesian Nash Equilibrium

We have already seen that a strategy for a player in a game of incomplete information is a function that specifies what action or actions to take in the game, for every possible type of that player.

A *Bayesian Nash Equilibrium* is a Nash equilibrium of this game (in which the strategy set is the set of action functions).

There are two ways of finding a pure-strategy Bayesian Nash Equilibrium (BNE).

**Method 1.** This method works directly on the Bayesian normal form representation, which is most easily done by converting the game into the corresponding payoff matrix. Simply find the Nash equilibria from the payoff matrix.
This method computes expected payoffs from an *ex ante* perspective, before the players learn their types. Notice that, if the set of actions available or the set of possible types is infinite, we cannot construct the payoff matrix so method 1 will not work.

Here is the payoff matrix for the Entry Game with Cost Uncertainty, with best responses marked with a star.

<table>
<thead>
<tr>
<th></th>
<th>firm 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><em>E</em></td>
<td><em>N</em></td>
</tr>
<tr>
<td><em>E</em></td>
<td>−(\frac{1}{4}, \frac{1}{4})</td>
<td>(\frac{3}{4}, 0)*</td>
</tr>
<tr>
<td><em>L</em></td>
<td>0* , (\frac{1}{4})</td>
<td>(\frac{1}{2}, 0)</td>
</tr>
<tr>
<td><em>N</em></td>
<td>−(\frac{1}{4}, \frac{1}{4})</td>
<td>(\frac{1}{4}, 0)</td>
</tr>
<tr>
<td></td>
<td>0* , (\frac{3}{4})</td>
<td>0 , 0</td>
</tr>
</tbody>
</table>

There are three BNE: \((E^L E^H, N)\), \((N^L N^H, E)\), and \((E^L N^H, E)\).
Here is the payoff matrix for the Gift Game, with best responses marked with a star for the case in which player 1 is more likely to be a friend $p > \frac{1}{2}$.

<table>
<thead>
<tr>
<th>player 1</th>
<th>player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N^F N^E$</td>
<td>$A$</td>
</tr>
<tr>
<td></td>
<td>0, 0*</td>
</tr>
<tr>
<td>$N^F G^E$</td>
<td>1 − $p$, $p − 1$</td>
</tr>
<tr>
<td>$G^F N^E$</td>
<td>$p$, p*</td>
</tr>
<tr>
<td>$G^F G^E$</td>
<td>1*, 2$p − 1$*</td>
</tr>
</tbody>
</table>

Thus, when $p > \frac{1}{2}$, the BNE are $(N^F N^E, R)$ and $(G^F G^E, A)$. 
Here is the payoff matrix for the Gift Game, with best responses marked with a double star for the case in which player 1 is more likely to be an enemy $p < \frac{1}{2}$.

\[
\begin{array}{c|cc|c|cc}
 & \text{player 2} & & & & \\
 & A & & & & \\
\hline
\text{player 1} & NFNE & & & & \\
& NFGE & & & & \\
& GFNE & & & & \\
& GFGE & & & & \\
\hline
NFNE & 0, 0** & & & & 0**, 0** \\
NFGE & 1 - p, p - 1 & & & & p - 1, 0** \\
GFNE & p, p** & & & & -p, 0 \\
GFGE & 1**, 2p - 1 & & & & -1, 0** \\
\end{array}
\]

Now the game has only one BNE, $(NFNE, R)$. No matter what $p$ is, it is an equilibrium for player 1 never to offer because the gift will be refused. There is also a BNE in which player 2 accepts and both types of player 1 offer, but only if type F is more likely than type E.
Method 2. This method for finding the BNE converts the game into an equivalent "bigger" game in which the different types of each player are treated as separate players. The payoff to player \((i, t_i)\) is the expected payoff of player \(i\), conditional on being type \(t_i\).

Any NE of the bigger game is a BNE of the original game, and vice versa:

The payoff in the Bayesian normal-form matrix is the summation over all types of the probability of a type multiplied by the expected payoff conditional on that type.

\[ \text{If player } i \text{ is best responding in the original game, then it is impossible to increase his payoff conditional on any of his types (or else this summation would be higher), so each player } (i, t_i) \text{ must be best responding in the bigger game.} \]
If each player \((i, t_i)\) is best responding in the bigger game, then it is impossible to increase his ex ante expected payoff in the original game by choosing a different strategy function \(a_i(t_i)\). (The only way to achieve a higher summation of probability of \(t_i\) multiplied by utility conditional on \(t_i\) is to increase at least one term in the sum, which is impossible if each player \((i, t_i)\) is best responding in the bigger game.)

Method 2 is often easier than Method 1, especially when players have an infinite number of possible actions (Cournot game with cost uncertainty) or an infinite number of types (auction with a continuous distribution of possible valuations for the object being auctioned).

Let us apply Method 2 to the Entry Game with Cost Uncertainty, and see that we get the same answer as in Method 1.
In the bigger game, there are three players, $(1, H)$, $(1, L)$, and 2. Let us find the NE of this game.

Can there be a NE in which firm 2 chooses $E$? Player $(1, H)$ receives a payoff of 0 by choosing $N^H$, and a payoff of $-\frac{1}{2}$ from choosing $E^H$, so the best response is $N^H$. Player $(1, L)$ receives a payoff of 0 by choosing $N^L$, and a payoff of 0 from choosing $E^L$, so both $N^L$ and $E^L$ are best responses.

Firm 2’s choice of $E$ is the best response to the profile $(N^L, N^H)$ [because $\frac{3}{4} > 0$] and it is a best response to the profile $(E^L, N^H)$ [because $\frac{1}{4} > 0$], so both $(N^L, N^H, E)$, and $(E^L, N^H, E)$ are NE of the 3-player game. Therefore, $(N^L N^H, E)$, and $(E^L N^H, E)$ are BNE of the Bayesian normal-form game.
Can there be a NE in which firm 2 chooses $N$? Player $(1, H)$ receives a payoff of 0 by choosing $N^H$, and a payoff of $\frac{1}{2}$ from choosing $E^H$, so the best response is $E^H$. Player $(1, L)$ receives a payoff of 0 by choosing $N^L$, and a payoff of 1 from choosing $E^L$, so the best response is $E^L$.

Firm 2’s best response to the profile $(E^L, E^H)$ is $N$ [because $0 > -\frac{1}{4}$]. Therefore, $(E^L, E^H, N)$ is a NE of the 3-player game, and $(E^L E^H, N)$ is a BNE of the Bayesian normal-form game.
Cournot Competition with Cost Uncertainty

Consider a (simultaneous move) Cournot game with the inverse demand function

\[ p = 1 - q_1 - q_2. \]

Firm 1's production cost is zero.

Firm 2 has two possible types, \( L \) and \( H \), each of which occur with probability \( \frac{1}{2} \).

A type \( L \) firm 2 has low marginal cost, 0, and a type \( H \) firm 2 has high marginal cost, \( \frac{1}{4} \).

In the Bayesian normal-form game, a strategy for firm 1 is a quantity, \( q_1 \), and a strategy for firm 2 is a function that specifies a quantity for each type, \( (q_2^L, q_2^H) \).
Since every nonnegative quantity is a possible strategy for firm 1 and every pair of nonnegative quantities is a possible strategy for firm 2, the resulting payoff matrix would have an infinite number of rows and columns! Clearly Method 1 will not be easy.

Under Method 2, we consider the three player game with firm 1, firm 2L, and firm 2H. To find the NE, we compute the best response functions for all three players and solve the three equations for the three NE quantities.

Starting with firm 2L, its payoff function is

\[ u_2^L = (1 - q_1 - q_2^L)q_2^L. \]

Differentiating with respect to \( q_2^L \), setting the expression equal to zero, and solving for \( q_2^L \), we can solve for firm 2L’s best response function.

\[
\frac{\partial u_2^L}{\partial q_2^L} = 0 = 1 - q_1 - 2q_2^L
\]

\[
BR_2^L(q_1) = \frac{1 - q_1}{2}.
\]
The payoff function of firm $2H$ is given by

$$u^H_2 = (1 - q_1 - q^H_2)q^H_2 - \frac{q^H_2}{4}.$$  

Differentiating with respect to $q^H_2$, setting the expression equal to zero, and solving for $q^H_2$, we can solve for firm $2H$’s best response function.

$$\frac{\partial u^H_2}{\partial q^H_2} = 0 = 1 - q_1 - 2q^H_2 - \frac{1}{4}$$

$$BR^H_2(q_1) = \frac{3}{8} - \frac{q_1}{2}.$$
The payoff function for firm 1 is based on the expectation that half of the time it is competing with firm 2_L and half of the time it is competing with firm 2_H.

\[ u_1 = \frac{1}{2}[(1 - q_1 - q_2^L)q_1] + \frac{1}{2}[(1 - q_1 - q_2^H)q_1] \]
\[ = (1 - q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2})q_1 \]

Differentiating with respect to \( q_1 \), setting the expression equal to zero, and solving for \( q_1 \), we can solve for firm 1’s best response function.

\[ \frac{\partial u_1}{\partial q_1} = 0 = 1 - 2q_1 - \frac{q_2^L}{2} - \frac{q_2^H}{2} \]
\[ BR_1(q_2^L, q_2^H) = \frac{1}{2} - \frac{q_2^L}{4} - \frac{q_2^H}{4}. \]
The Nash equilibrium is the solution to the following three equations

\[ BR^L_2(q_1) = \frac{1}{2} - \frac{q_1}{2} = q_2^L \]

\[ BR^H_2(q_1) = \frac{3}{8} - \frac{q_1}{2} = q_2^H \]

\[ BR_1(q_2^L, q_2^H) = \frac{1}{2} - \frac{q_2^L}{4} - \frac{q_2^H}{4} = q_1 \]

To solve, substitute \( q_2^L \) from the first equation and \( q_2^H \) from the second equation into the third equation, and solve for \( q_1 \).

\[ \frac{1}{2} - \left( \frac{1}{8} - \frac{q_1}{8} \right) - \left( \frac{3}{32} - \frac{q_1}{8} \right) = q_1 \]

\[ \frac{9}{32} + \frac{q_1}{4} = q_1 \]

\[ q_1 = \frac{4}{3} \left( \frac{9}{32} \right) = \frac{3}{8}. \]
Substituting $q_1 = \frac{3}{8}$ into the remaining two equations, we have the Nash equilibrium strategy profile for this three player game, $q_1 = \frac{3}{8}, q_2^L = \frac{5}{16}, q_2^H = \frac{3}{16}$.

Thus, the BNE for the original game with two players is the following:

$$q_1 = \frac{3}{8}, \quad (q_2^L, q_2^H) = \left( \frac{5}{16}, \frac{3}{16} \right).$$

For this game, firm 1’s quantity is a best response to the average quantity selected by firm 2.
Auction Markets

We will solve the games corresponding to various auction rules under the following environment:

There is one indivisible object being sold.

The players are the \( n \) bidders. Each player has a valuation for the object, with player \( i \)'s valuation denoted by \( v_i \).

If player \( i \) wins the auction and makes a payment, \( p \), her overall payoff is \( v_i - p \); if she does not win the auction but she makes a payment, \( p \), her overall payoff is \(-p\).

We assume that each \( v_i \) is independently drawn from the uniform distribution over the unit interval \([0, 1]\). In other words, all realizations between 0 and 1 are equally likely, and knowing \( v_i \) provides no information about the other players’ valuations.

The players simultaneously submit bids, where the bid of player $i$ is denoted by $b_i$. Each player observes her valuation (her type) before deciding what to bid, so a strategy is a bid function, $b_i(v_i)$.

The player submitting the highest bid wins the auction and makes a payment equal to her bid.

Players who do not win the auction do not make a payment; their payoff is zero. In case of a tie for the highest bid, someone is randomly selected as the winner.

What would you bid if $n = 10$ and your valuation is 0.6?
Notice that it does not make sense to bid more than your valuation, because your payoff cannot be positive. Either you lose the auction and receive zero, or win the auction and receive a negative payoff.

In fact, players should bid less than their valuation so that if they win the auction, their payoff is positive.

Finding the BNE is not easy. We will guess that there is a symmetric BNE in which all players bid a constant fraction of their valuation:

\[ b_i(v_i) = av_i \]

for some number \( a \) that is the same for all players and is between zero and one.

Then we will use the condition that every type of every player is best-responding to the other players by bidding this way. This will allow us to solve for the value of \( a \) that make this symmetric profile of bidding functions a BNE.
Consider player $j$ with valuation $v_j$, and suppose that all of the other players are bidding according to $b_i(v_i) = av_i$. Then the highest possible bid by one of the other players (with $v_i = 1$) is $a$, so there is no reason to bid more than $a$.

If player $j$ makes a bid of $b$, she wins the auction if and only if all of the other bids are below $b$. Based on their bidding functions, this happens if we have for all $i \neq j$,

$$av_i < b, \text{ or } v_i < \frac{b}{a}.$$
Because of the uniform distribution, the probability that a particular one of the other players has a valuation below $\frac{b}{a}$ is $\frac{b}{a}$.

Then the probability that all of the other players have valuations below $\frac{b}{a}$, so that player $j$ wins the auction, is given by

$$pr(j \text{ wins when bidding } b) = \left(\frac{b}{a}\right)^{n-1}.$$ 

This allows us to express player $j$’s payoff as a function of her valuation and her bid, given the bidding functions of the other players.

$$u_j = \left(\frac{b}{a}\right)^{n-1} [v_j - b]$$
We can now find the optimal bid for each type of player $j$, by taking the derivative of $u_j$ with respect to $b$, setting the expression equal to zero, and solving for $b$.

\[a^{1-n}[(n-1)b^{n-2}(v_j - b) + b^{n-1}(-1)] = 0\]

\[(n-1)(v_j - b) - b = 0\]

\[(n-1)v_j - nb = 0\]

\[b = \frac{n-1}{n}v_j\]

Thus, if other players are bidding a constant fraction of their valuations (no matter what the constant, $a$), the best response of bidder $j$ is to bid a constant fraction, $\frac{n-1}{n}$, of her valuation.

Therefore, we have a BNE if each player $i$ uses the bidding function,

\[b_i(v_i) = \frac{n-1}{n}v_i.\]
Recapping, in the sealed-bid first-price auction, the BNE is for all players to use the bidding function, $b_i(v_i) = \frac{n-1}{n} v_i$.

Players shade their bid below their valuation, to balance the profit when they win against the risk of not winning. The more players in the auction, the less they can afford to shade their bid.

The BNE outcome is efficient, because the player with the highest valuation always wins the auction.

When $n = 2$, players bid half their valuation, so when the highest valuation is $\overline{v}$, the player with that valuation receives a payoff of $\frac{\overline{v}}{2}$ and the seller receives revenue of $\frac{\overline{v}}{2}$.
2. **Sealed-bid, second-price auction.**

The players simultaneously submit bids, where the bid of player $i$ is denoted by $b_i$. Each player observes her valuation (her type) before deciding what to bid, so a strategy is a bid function, $b_i(v_i)$.

The player submitting the highest bid wins the auction and makes a payment equal to the second highest bid.

Players who do not win the auction do not make a payment; their payoff is zero. In case of a tie for the highest bid, someone is randomly selected as the winner.

For example, if player 3 submits the highest bid of 0.78 and player 6 submits the second-highest bid of 0.62, then player 3 wins the auction (driving home with the object) and pays 0.62, while the other players do not receive the object or make any payments.

What would you bid if $n = 10$ and your valuation is 0.6?
One could try to solve for the BNE of the sealed-bid second-price auction the same way that we solved the first-price auction, but there is a much easier way.

Notice that it is a weakly dominant strategy to bid your valuation:

If the highest of the other players’ bids (call it $v'$) is greater than $v_i$, then bidding $v_i$ is a best response. Player $i$ loses, but changing her bid in order to win would require her to bid more than $v'$, in which case she would pay $v'$, which is more than her valuation.

If $v' < v_i$, then bidding $v_i$ is a best response. Player $i$ wins the auction and receives a positive payoff. Changing her bid while still winning does not change her payment, $v'$, and changing her bid to something below $v'$ reduces her payoff to zero.
The symmetric BNE has all players choosing the bidding strategy $b_i(v_i) = v_i$.

The BNE outcome is efficient, because the player with the highest valuation always wins the auction.

When $n = 2$, since players bid their valuation, when the highest valuation is $\bar{v}$, the player with that valuation wins the auction and makes a payment equal to the other player’s valuation. The payment is uniformly distributed over the interval from 0 to $\bar{v}$, so the expected payment is $\frac{\bar{v}}{2}$. Thus, the winner receives an expected payoff of $\frac{\bar{v}}{2}$ and the seller receives expected revenue of $\frac{\bar{v}}{2}$.

The expected payoff to the players and the expected revenue to the seller is the same as in the first-price auction. This result that payoffs do not depend on the auction format (called revenue equivalence) extends to $n$ players and to valuation distributions other than uniform.
These sealed-bid auctions have dynamic counterparts.

The (independent private values) second-price sealed bid auction is essentially equivalent to the ascending-price or English auction you see in movies or on eBay.

The (independent private values) first-price sealed bid auction is essentially equivalent to the descending-price or Dutch auction.

The game changes significantly if payoffs have a "common value" component. That is, if the object’s worth to me increases when I learn that the object is worth a lot to you.

For example, if we are auctioning two tickets to the Buckeye game on the 50 yard line in the first row of C-deck, then the bidders know their valuations, which would not change if they learned that the other bidders had high valuations. If instead the seat locations are not specified and bidders’ types in part reflect information about where the seats are, then learning your type tells me something about what the seats would be worth to me.